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The Husimi Distribution of Circular Billiard with an Applied Uniform Magnetic Field

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Abstract We present a theoretical computation of the Husimi distribution function in phase-space for studying the semiclassical dynamics of the circular electron billiard subjected to a constant magnetic field in the perpendicular direction. The results reveal that with the increase of the applied magnetic field the peaks of Husimi function tend to the billiard boundaries, along with the movements a periodic splitting-recombining (alternative single-double) peak structure is arisen. This fact implies the localization of the eigenstates and coincides to the classical trajectory distribution what we obtained by use of representation on the billiard boundary. It becomes possible to compare the local properties of the quantum and classical distributions. Our analysis provides a new perspective to understand the quantum-classical correspondence.

Keywords Husimi distribution · Circular billiard · Classical trajectory · Effect of external magnetic field

1 Instruction

The quantum-classical correspondence for complicated systems is a subject of considerable current interest and has achieved great developments in concept sense. Due to lack of a direct connection, however, seeking distributions that allow for a common view of classical and quantum mechanics is still unclear even for some integrable systems. In recent years most efforts are devoted to the exploration of the quantum properties of classical chaotic

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systems particularly to the statistic properties of the spectrum [1–6], for instance, the Liouville dynamics method [7, 8], the analysis via finite-time Lyapunov exponents [9]. Another attempt to improve theoretical approach also has been proposed [10]. By choosing to deal with the phase-space representation of quantum mechanics one can then compare the classical and quantum dynamics of distributions in phase-space. The phase-space distribution function introduced by Wigner [11] in 1932 offers a convenient framework in which quantum phenomena can be described by using as much classical language as possible, it can often provide useful physical insights that cannot easily be gained by other way. Furthermore, it needs only deal with finite-number equations and avoidable to directly face on the problem of inconvenient operators, and often is a significant advantage. Therefore, the relevant methods have been applied to different fields [12], such as quantum optics [13], collision theory [14, 15], and nonlinear physics [16]. But few of study on the quantum billiards systems range from regular to chaotic characters are concerned. Billiard is one of the simplest and the most studied prototype, physically their dynamics involves widely classical and quantum mechanics, electromagnetic character, acoustic or optical progresses. In this paper as an example to show the semiclassical behaviors of the system, we will study the circular electron billiard with uniform magnetic field normal to the billiard plane. The importance is mainly related with the semiclassical magnetic susceptibility of an ensemble of noninteracting electrons at low temperatures. The connection with the quantum Hall effect is also evident. Based on the calculations of eigenvalues and eigenstates, the Husimi distribution functions can be obtained by smoothing the Wigner function via Gaussian function. Because the general Husimi function in phase-space may be related with the Poincaré Husimi function on the billiard boundaries and can be viewed as a probability density on the Poincaré surface of section [17], therefore, it turns easily out a direct connection with the classical trajectory distributions. It is becomes possible to compare the both distributions, this approach thus is helpful to understand quantum-classical correspondence.

2 The Husimi Distribution of the Circular Billiard Subjected to a Uniform Magnetic Field in the Phase-Space

We consider a two dimensional circular electron billiard in applied perpendicular uniform magnetic field with the radius $r = 1$ in atomic units. The Hamiltonian of an electron in the magnetic field can be written as

$$H = \frac{1}{2M}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 \tag{1}$$

For sake of convenience, in this paper we shall work with the symmetric gauge $\vec{A}(\vec{r}) = \frac{1}{2}\vec{B} \times \vec{r}$, that is $A_r = 0$ and $A_\varphi = \frac{1}{2}Br$ in the polar coordinates. Then the Schrödinger equation is expressed as the following

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{ieB}{\hbar c}\frac{\partial\psi}{\partial\varphi} - \left(\frac{eBr}{2\hbar c}\right)^2\psi + \frac{2M\varepsilon}{\hbar^2}\psi = 0 \tag{2}$$

The wave function ψ has a variable separated form

$$\psi(r, \varphi) = R(\xi)\,\Phi(\varphi) \tag{3}$$

and the variable ξ is related to the radial coordinate r

$$\xi = \frac{eBr^2}{2\hbar c} \tag{4}$$

the relevant wave function will meet the below equation

$$\xi \frac{d^2R}{d\xi^2} + \frac{dR}{d\xi} + \left[\frac{\varepsilon}{\hbar\omega_c} - \frac{q^2\hbar}{2M\omega_c} + \frac{1}{2}l - \frac{1}{2}\xi - \frac{l^2}{4\xi}\right] R = 0 \tag{5}$$

with ω_c is the cyclotron frequency, $\omega_c = \frac{eB}{Mc}$. In order to solve the above equation, we should concretely consider the asymptotic behaviors of ξ, that is

$$R(\xi) \underset{\xi\to\infty}{\to} \exp\left(-\frac{\xi}{2}\right) \tag{6}$$

$$R(\xi) \underset{\xi\to 0}{\to} \xi^{\frac{1}{2}} \tag{7}$$

the wave function can then be expressed as

$$R(\xi) = \exp\left(-\frac{\xi}{2}\right)\xi^{\frac{1}{2}}\chi(\xi) \tag{8}$$

where the $\chi(\xi)$ satisfies the following standard confluent hypergeometric equation

$$\xi\left(\frac{d^2\chi}{d\xi^2}\right) + (l + 1 - \xi)\frac{d\chi}{d\xi} + \alpha\chi = 0 \tag{9}$$

The parameter $\alpha = \frac{\varepsilon-(l+|l|+1)}{2\hbar\omega}$ is related with the eigenenergy ε, thus the energy is

$$\varepsilon = 2\hbar\omega\left(\alpha + \frac{l + |l|}{2} + \frac{1}{2}\right) \tag{10}$$

The solution of (9), the confluence hypergeometric function can be expanded as an infinite series

$$F(-\alpha, |l+1|, \xi) = 1 + \frac{(-\alpha)}{(l+1)}\xi + \frac{(-\alpha)(-\alpha+1)}{(l+1)(l+2)}\frac{\xi^2}{2!} + \cdots \tag{11}$$

For a circular quantum billiard in the magnetic field, the value of the parameter α is determined by the circle boundary condition

$$\chi(\xi_\alpha) = F(-\alpha, |l| + 1, \xi_\alpha) = 0 \tag{12}$$

Obviously, the argument is $\xi_\alpha = \frac{eBa^2}{2\hbar c}$.

Therefore the eigenfunction of the circular quantum billiard can be expressed as

$$\psi_\pm(r,\varphi) = \exp(\pm il\varphi)\exp\left(-\frac{\xi}{2}\right)\xi^{\frac{1}{2}}F(-\alpha_l, |l| + 1, \xi) \tag{13}$$

where $l = 0, 1, 2, \ldots$, it can be written alternatively in other form

$$\psi(r,\varphi) = \exp(il\varphi)\exp\left(-\frac{\xi}{2}\right)\xi^{\frac{1}{2}}F(-\alpha_l, |l| + 1, \xi)$$

for $l = 0, \pm 1, \pm 2, \ldots$.

The Wigner distribution function is defined for a quantum system

$$F^w(q,p,t) = \frac{1}{4\pi^2}\int d\xi d\eta\, \mathrm{Tr}(\hat{\rho}(\hat{q},\hat{p},t)e^{i\xi\hat{q}+i\eta\hat{p}})e^{-i\xi\hat{q}-i\eta\hat{p}}$$
$$= \frac{1}{2\pi}\int d\eta \left\langle q+\frac{1}{2}\eta\hbar|\hat{\rho}|q-\frac{1}{2}\eta\hbar\right\rangle e^{-i\eta p} \quad (14)$$

For the pure states, insert the $\hat{\rho} = |\psi\rangle\langle\psi|$ into the above formula, and let $x = \frac{\eta\hbar}{2}$, (14) becomes

$$F^w(q,p,t) = \frac{1}{\pi\hbar}\int dx \exp\left(\frac{-2ipx}{\hbar}\right)\psi * (q-x,t)\,\psi\,(q+x,t) \quad (15)$$

the integration is performed in all area of the billiard.

Although the Wigner distribution function is available and projects onto the correct marginal distributions, however it is incapable to deal with some complicated systems, particularly for the highly degenerate situations because of there are negative definite in many regions of the phase space. Moreover, some simple examples show that the Wigner function oscillates violently in classical limitation [18, 19]. But Husimi function (15) via a Gaussian smoothing would guarantee the corresponding matrices always to be positive definite. The method is useful also because it relates intimately with the coherent state describing the electron-magnetic fields [20, 21], which have been shown to be the best approximation of the classical representation of the fields [13]. Then the Husimi function constructs a quantum resonant version and can be defined as

$$F^H(q,p,t) = \frac{1}{\pi\hbar}\int dq'dp' \exp\left[\frac{-mk(q'-q)^2}{\hbar} - \frac{(p'-p)^2}{\hbar mk}\right]F^w\left(q',p',t\right) \quad (16)$$

where k is an arbitrary positive constant that determines the full width at the half maximum (FWHM).

In our calculations the atomic units have been used that is the electron mass $m = 1$, the electron charge $e = 1$, the strength of the magnetic field B ranges from 0.3 to 3.5 atomic unite. To discuss the correspondence between the quantum and classical mechanics, the classical limit requires that the Plank constant should be as small as possible, for instance, we have took $\hbar = 0.05$. The area of the billiard region is π (the radius is 1). In Fig. 1, we show the Husimi distribution with variation of the applied magnetic field from $B = 0.3, 1.9, 2.3$ to 3.5 that are denoted in plates of a $\sim$ d, respectively. When the magnetic field is 0.3 the Husimi distribution (HD) has a single peak at the middle of the phase space. As the magnetic field is increased to 1.9, the peak of HD splits into two peaks in the q direction while the peak location moves toward the billiard boundaries, the magnetic field is consequently increased to 2.3, the peak of HD is close to the billiard boundaries, and the plot returns to a single peak along with a vale appears in the p increasing direction. As the magnetic field is further increased to 3.5, the peak of HD splits into two peaks also in the p direction. Since the HD peaks stand for the localization of the electronic distribution in the cavity. With the magnetic field increasing, the localization moves to the billiard boundaries and appears more than one localized position. The Husimi distribution has widely used in the quantum mechanics as a counterpart of the classical Poincaré surfaces of section. Because the general Husimi function in phase space may be related to the Poincaré Husimi function

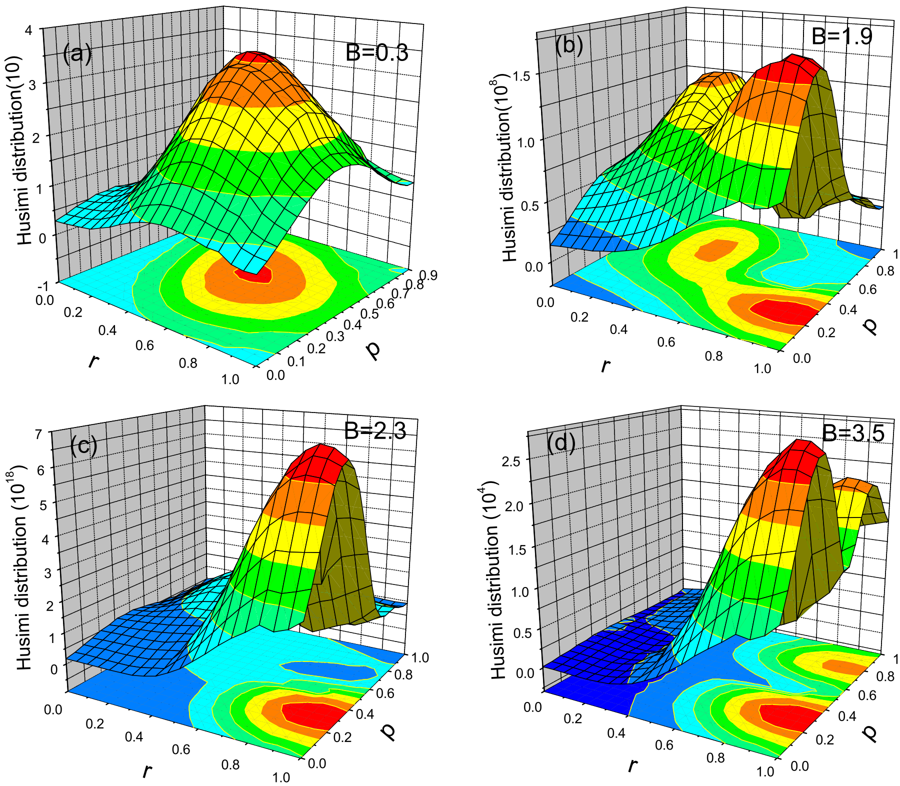

Fig. 1 (Color online) The Husimi distributions with variation of the magnetic field from $B = 0.3, 1.9, 2.3$ to 3.5 in atomic units. The *upper part* of every graph is the three-dimension Husimi distribution and the *bottom part* is the contour plot projected onto the phase-space (two-dimensional plane). The color of the contour plot corresponds to three-dimension Husimi distribution. For example the *shaded red areas* corresponding to the peaks of the Husimi function denotes the maximum of the state population and the *blue areas* corresponding to the vale of the Husimi function denotes the region electron cannot appear

on the billiard boundaries and can be viewed as a probability density on the Poincaré section [22], therefore, it turns out a direct connection with the classical trajectory distributions. For integrable systems the peaks of the Husimi distribution correspond to the points that the classical periodic trajectories through the surface of the section [22].

3 The Classical Trajectory in the Framework of the Representation on the Billiard Boundary

We choose the scale of the system in interval between the Fermi wave length and the average free path of the electron. The interaction of electrons can be ignored, and the electrons can be treated as free particles. The motion of the electrons obeys the classical dynamics, then will be elastically reflected when they reach the billiard boundaries, and the force that makes the electrons to move inside billiard is induced by the applied constant magnetic field

$$\vec{F} = q\vec{v} \times \vec{B} = q\left(\dot{x}\vec{i} + \dot{y}\vec{j}\right) \times B\vec{k} = qB\dot{y}\vec{i} - qB\dot{x}\vec{j} \tag{17}$$

the dynamical motion equations are

$$
\begin{aligned}
m\ddot{x} &= qB\dot{y} \\
m\ddot{y} &= -qB\dot{x}
\end{aligned}
\tag{18}
$$

The corresponding boundary and initial conditions are provided, respectively

$$
\begin{aligned}
&x = 0, \qquad y = 0 \\
&\dot{x} = v_{0x}, \qquad \dot{y} = v_{0y} \\
&\dot{x} = \frac{qB}{m} y + v_{0x}
\end{aligned}
$$

Then

$$
\ddot{y} = -\left(\frac{qB}{m}\right)^2 y - \frac{qB}{m} v_{0x} \tag{19}
$$

We notice that the above (19) is a differential equation for simple harmonic oscillation, its solution can be given easily

$$
\begin{aligned}
y &= A \sin(\omega t + \alpha) - \frac{v_{0x}}{\omega} \\
x &= -A[\cos(\omega t + \alpha) - \cos(\alpha)]
\end{aligned}
\tag{20}
$$

with

$$
\omega = \frac{qB}{m}, \qquad A = \frac{1}{\omega}\sqrt{v_{0x}^2 + v_{0y}^2} = \frac{v'}{\omega}, \qquad \cos(\alpha) = \frac{v_{0y}}{v'}
$$

Therefore the classical trajectory of the electron moving in the cavity can be rewritten as

$$
\begin{aligned}
x &= -\frac{1}{\omega} v' [\cos(\omega t + \alpha) - \cos(\alpha)] + x_0 \\
y &= \frac{1}{\omega} v' \cos(\omega t + \alpha) - \frac{v_{0x}}{\omega} + y_0
\end{aligned}
\tag{21}
$$

Where the (x_0, y_0) is the initial point.

In Fig. 2, we have plotted some classical trajectories with the cyclotron radius $r = \frac{\sqrt{2E}}{B}$, the representation on the billiard boundary is utilized here. (Does not loss of the generality, the starting points of the electron motion are assumed at arbitrary boundary points. The parameters are shown in Table 1.) Classical trajectory family often denotes the density that the electrons appear in the configuration space. Therefore the classical trajectory family can be defined the density of localization of the particles. In Fig. 2(a) and (c), with the increase of the magnetic field, the localized position where the classical trajectories are concentrated, toward the boundaries and there are multiple maximums and minimums, which is consistent with the result of the Husimi distribution.

4 Summary

The aim of this paper is to explore and develop the Husimi function distribution as a recipe for representing the state distribution in phase-space of quantum billiard system in a uniform

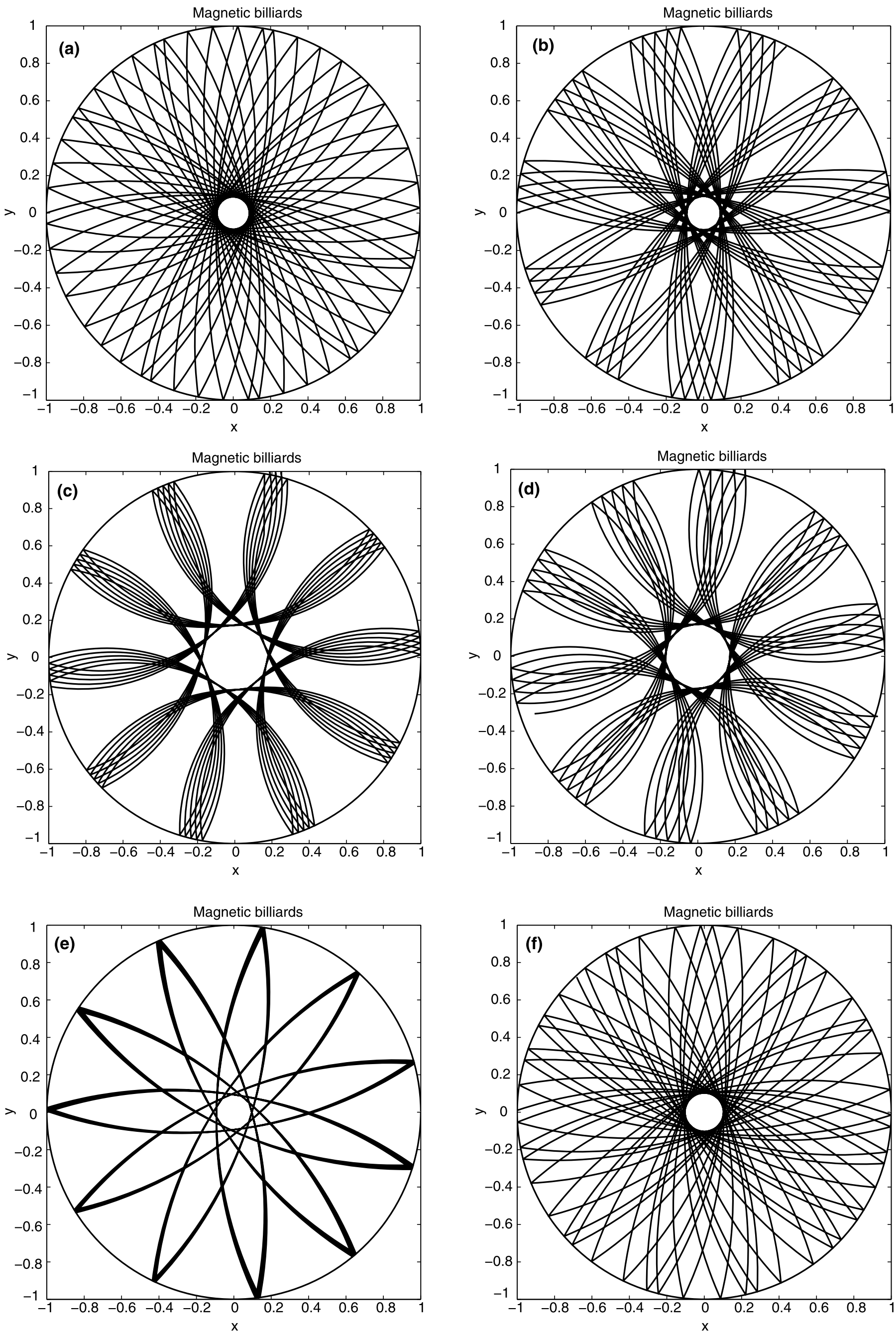

Fig. 2 The classical trajectory family of electron motion in the circular billiard with the uniform magnetic field. (Does not loss of the generality, the initial motion of the electron is assumed is starting from an arbitrary boundary point.) The initial directions of the motions in plates of (**a**) and (**c**) are distinct from that in plates of (**b**) and (**d**). In plates of (**e**) and (**f**) the electron movies alone curves with different radius r of the arc, while with same motion direction and different initial y coordinates

Table 1 The parameters correspond to Fig. 2

	Figure 2(a)	Figure 2(b)	Figure 2(c)	Figure 2(d)	Figure 2(e)	Figure 2(f)
Cyclotron radius r	2.2	2.2	1.0	1.0	2.2	2.2
Initial position (x, y)	$(-1.0, 0.0)$	$(-1.0, 0.0)$	$(-1.0, 0.0)$	$(-1.0, 0.0)$	$(-1.0, 0.0)$	$(-1.0, 0.01)$
Motion direction (θ)	0.101π	0.102π	0.101π	0.102π	0.103π	0.103π

magnetic field, and analyze the correspondence of the quantum and classical mechanics. We emphasize that the circular billiard subjected to a uniform magnetic field is integrable system. The dynamics equation can be exactly solved to obtain the wave function. So the Husimi distribution (HD) can be calculated. From the numerical results, we find that there are evidently localizations of the wave function of the electron which corresponds to the concentration of the classical trajectory family. When the magnetic field is increased, the position of the localization moves to the verge of the billiard domain. Along with the movements a periodic splitting-reviving (alternative single-double) peak structure is obviously arisen. We speculate that if the geometric boundaries of a billiard do not coincide with the magnetic cyclotron circle, the multiple scattering of the electron happens at the boundaries and the dynamic behaviors will give rise to chaotic.

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